11 –Optimization

Optimization is a subject that not only could be an entire course but an entire degree program. It is the province of Operations Research. Nevertheless, engineers and scientists frequently need optimization tools to find optimal solutions to multivariate problems including the efficient allocation of resources to achieve a maximized output or minimized cost, the best coefficients for a curve fit of measured data, and the adjusting of data in a table to the best values that continue to observe known, constraining relationships among the data. A few tools are presented here that have been selected for their usefulness to all engineering and science professionals. They are
- Linear Programming: Simplex, Graphical, Big-M
- Microsoft Excel Tools: Goal Seek and Solver
- Data Adjustment
- Regression Analysis

11.1 –Introduction to Linear Programming
Linear programming is a relatively recent optimization method. During World War II, Dantzig developed the method's fundamentals and it was adopted quickly as an efficient way of solving a wide range of problems encountered in engineering and other disciplines. In engineering, linear programming is used to solve problems involving personnel assignment, building design, transportation, water resource management, and countless other areas.

Linear programming problems involve the allocation of limited resources to the production of alternative products or services. For example, an electronics firm that produces calculators, iPods, and PDAs is limited in its total productivity by the maximum number of employees it can locate in its plant and equip with tools. At full production the firm has 1000 employee-hours per day for the production of the three items. The firm may use linear programming to determine how to allocate its work force so as to maximize the profit derived from the production of its three items.

The solution of this problem by linear programming requires that all constraints that limit the production of each item be known. For example, previous agreement to another business may require that the firm make at least five iPods per day. On the other hand, license agreements may restrict the maximum number of calculators per day to ten. In addition to establishing the above constraints, the firm cannot use linear programming to solve this problem unless a linear relationship exists between the number of employee hours and the number of each item produced. That is, each iPod, produced must require the same number of employee-hours whether it is the first or last iPod, produced. Even though a different number of employee-hours may be required to produce a PDA, than a calculator, the same numbers of hours must be required for each PDA, and each calculator. In addition to the linear relationship between inputs (employee-hours, soldering facility hours, license agreements) and outputs (calculators, PDAs, iPod), a linear relationship must also exist between the profit derived from each item and the number of each item produced. These characteristics of linear programming problems can be stated in a precise, mathematical manner.

Linear Output Constraints: If the output quantities are x_1, x_2, x_3, then

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n \leq b_1 \tag{11.1}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \leq b_2 \tag{11.2}$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots a_{in}x_n \leq b_i \tag{11.3}$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mN}x_N \leq b_m \tag{11.4}$$

where   N =   Number of variables
           m =   Number of constraining equations
           a =   Constraint constants
           b =   Limitations in available resources.

This matrix of equations is not difficult to understand in a particular example. In the case of the electronics firm, one constraint equation comes from the fact that no more than 1000 employee-hours are available per day for production. Therefore,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq 1000 \tag{11.5}$$

where $x_1 =$ Number of iPods produced per day
           $x_2 =$ Number of PDAs produced per day
           $x_3 =$ Number of calculators produced per day
           $a_{11} =$ The number of employee-hours required to
                      produce one iPod
           $a_{12} =$ The number of employee-hours required to
                      produce one PDA
           $a_{13} =$ The number of employee-hours to produce
                      one calculator

If the firm is not allowed by an existing license agreement to make more than ten calculators a day, then

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq 10 \tag{11.6}$$

where $a_{21} = a_{22} = 0$ and $a_{23} = 1$.

If the circuit boards for both the iPods and the PDAs must be soldered on the firm's only automatic soldering machine that is available only 16 hours per day, then

$$a_{31} + a_{32}x_2 + a_{33}x_3 \leq 16 \tag{11.7}$$

where $a_{31} =$ The number of hours it takes to solder each iPod
           $a_{32} =$ The number of hours it takes to solder each PDA
           $a_{33} =$ The number of hours it takes to solder each calculator

Linear Objective Function: The objective function, f, must be described mathematically as

$$f = \sum_{j=1}^{N} c_j x_j \tag{11.8}$$

where c_j = the objective function coefficient
           x_j = The number of j units of output

In the case under consideration, the objective is profit maximization and the objective function is

$$f = c_1x_1 + c_2x_2 + c_3x_3 \tag{11.9}$$

where c_1, c_2, and c_3 are the profits realized from the production of each iPod, PDA, and calculator.

Optimization problems, which can be put in the above form, may be solved by a variety of linear programming techniques. Problems that can be formulated with two unknowns can be solved by the Graphical Method. A variety of techniques have been developed for solving linear programming problems having as many as thousands of unknowns and thousands of constraint equations. Sophisticated code methods have been devised to solve such massive problems with a computer. The less-sophisticated, techniques called the Graphical Method and the Simplex Method are presented in the following section.

11.2 –Linear Programming Solutions by the Graphical Method

The graphical method of solving linear programming problems is limited to two variables since representation of more variables on a graph is too difficult to visualize. The graphical method consists of representing one output on the horizontal coordinate and the second output on the vertical coordinate. The constraints on each output are then plotted on the graph to define regions in which the solution cannot lie. The remaining region on the graph is called the *area of feasible solution*, and the objective function is used to find the optimum point within that region. The following example shows this procedure.

A civil engineer employed by a large contracting firm is in charge of construction projects in South Dakota. At present, three job sites within the State require a total of 120 tons of steel reinforcing bar. The steel was ordered by the district office and is deposited at Rapid City and Sioux Falls. There are 50 tons at Rapid City and 70 tons at Sioux Falls. The three job sites are at Pierre, Winner, and Faith. The Pierre job requires 60 tons, Winner 20 tons, and Faith the remaining 40 tons. Table 1 shows the transportation charges per ton of steel from Rapid City and Sioux Falls to each job site. The Graphical Method can be used to determine how the shipment of steel should be made to minimize the transportation cost.

Table 1. Transportation Charges in Dollars Per Ton

From /To	Pierre (60)	Faith (40)	Winner (20)
Rapid City (50)	3.0	2.0	1.80
Sioux Falls (70)	3.5	3.2	4.30

If x represents the tons of steel shipped from Rapid City to Pierre and y the tons shipped from Rapid City to Faith, then all the other tonnages may be expressed in terms of these variables as shown in the Table 2.

Table 2. Quantities of Steel Shipped

From /To	Pierre (60)	Faith (40)	Winner (20)
Rapid City (50)	x	y	50-x-y
Sioux Falls (70)	60-x	40-y	-30 + x + y

The amount of steel shipped from Rapid City to Winner is equal to the total 50 tons shipped from Rapid City minus the amounts shipped to Pierre and Faith, (x + y). Likewise, the quantities shipped from Sioux Falls to each site is determined by subtracting the amount shipped from Rapid City from the amount required at each site. If total supply exceeded total demand, a third variable for the amount shipped from Rapid City to Winner would be needed.

The objective function f in this problem is total shipping cost, which is the sum of the shipping costs to each city times the quantity shipped to each city.

$$f = 3.00x + 2.00y + 1.80(50\text{-}x\text{-}y) + 3.50(60\text{-}x) + 3.20(40\text{-} y) + 4.30 (x+ y\text{-}30) \qquad (11.10)$$

or

$$f = 299 + 2.0x + 1.3y \tag{11.11}$$

The constraints arise from the limitations in the quantities that can be shipped. None of the quantities in the table can be less than zero. Therefore,

x	≥ 0	(11.12)
y	≥ 0	(11.13)
50 - x – y	≥ 0	(11.14)
60 – x	≥ 0	(11.15)
40 – y	≥ 0	(11.16)
-30 + x + y ≥ 0		

Rearrangement gives

x	≥ 0	(11.17)
y	≥ 0	(11.18)
x + y	≤ 50	(11.19)
x	≤ 60	(11.20)
y	≤ 40	(11.21)
x + y	≥ 30	(11.22)

These inequalities are plotted in Figure 1. The unshaded areas are regions that violate the inequalities. The shaded area is called the *area of feasible solution*, because any point in the area constitutes an allowable solution (i.e. a solution that keeps all of the quantities shipped ≥ 0). The optimum solution within the *area of feasible solution* is determined using the objective function, f (x, y).

The cost equation can be rearranged to the form

$$y = \frac{f - 299}{1.3} - \frac{2.0}{1.3}x \tag{11.23}$$

This is a parametric equation and may be plotted providing a value of the parameter f is chosen. Figure 1 shows this equation for three values of f = 299, 319, and 361. These objective function lines all have the same slope of -2.0/1.3. The objective function line should be thought of as a sliding line with constant slope. The further the line moves into the first quadrant, the greater the shipping cost. Since the optimum solution must be the minimum cost *area of feasible solution*, the optimum solution is the first point in the *area of feasible solution* that the objective function line intercepts as it slides away from the origin towards the *area of feasible solution*. This point is (0, 30). Therefore, no steel should be shipped to Pierre from Rapid City while 30 tons should be shipped to Faith from Rapid City. Other tonnages may be determined by substituting x = 0 and y = 30 into the Table 2. The minimum cost is determined by substituting the optimal x and y values into the cost equation.

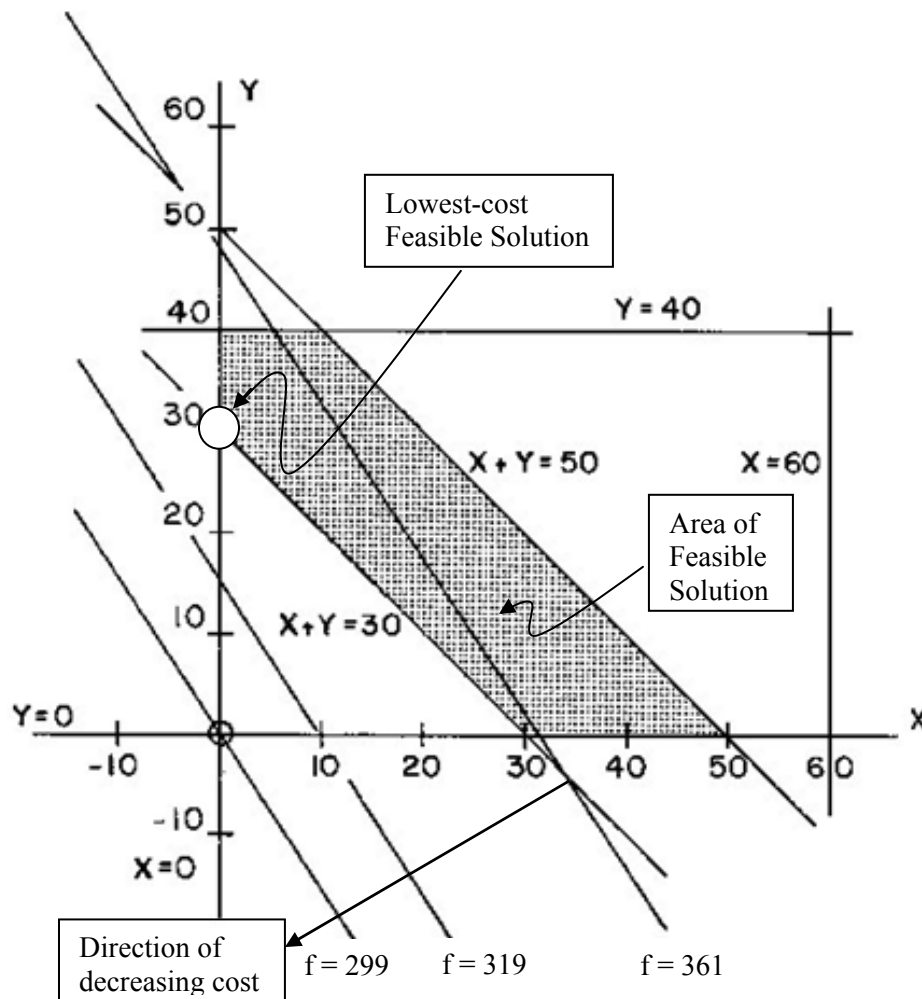$$f = 299 + 2.0(0) + 1.3(30) = \$338 \tag{11.24}$$

Figure 1. Area of Feasible Solution and Optimal Solution.

11.3 –Linear Programming Solutions by the Simplex Method

The Simplex Method of solving linear programming problems requires the manipulation of a set of equations obtained from the constraint equations and the objective function. The equations obtained from the constraints expressed by equations (11.1) through (11.4) are first written in terms of equalities by the introduction of a *slack* variable into each equation. No slack variable can be less than zero, because the Simplex Method solves only $\leq$ constraints. Each right-hand-side term represents a resource available, which can never be less than zero. Equations (11.1) through (11.4) then become

$$a_{11}x_1 + a_{12}x_2 + \ldots\ a_{1n}x_n + s_1\ =\ b_1 \tag{11.25}$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ a_{2n}x_n + s_2\ =\ b_2 \tag{11.26}$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots\ a_{in}x_n + s_i\ =\ b_i \tag{11.27}$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ a_{mn}x_n + s_m\ =\ b_m \tag{11.28}$$

where s_1, s_2, …s_i, s_m are the slack variables. The coefficients of these equations and the objective function then form the coefficient tableau shown in Tableau 1.

Tableau 1											
x_1	x_2	…	x_n	s_1	s_2		s_i		s_m	f	RHS
a_{11}	a_{12}	…	a_{1n}	1	0	…	0	…	0	0	b_1
a_{21}	a_{22}	…	a_{2n}	0	1	…	0	…	0	0	b_2
⋮	⋮	⋰	⋮	⋮	⋮	⋰	⋮	⋰	⋮	⋮	⋮
a_{i1}	a_{i2}	…	a_{in}	0	0	…	1		0	0	b_i
⋮	⋮	⋰	⋮	⋮	⋮	⋰	⋮	⋰	⋮	⋮	⋮
a_{m1}	a_{m2}	…	a_{mn}	0	0	…	0	…	1	0	b_m
$-c_1$	$-c_2$		$-c_n$	0	0	0	0	0	0	1	0

The objective function in the tableau is in the form

$$f - \sum_{i=1}^{m} c_i x_i = 0 \tag{11.29}$$

and the objective, f, is treated as an unknown.

Examination of Tableau 1 shows that excluding the last row there are m equations in m + N unknowns. Since there are N more unknowns than equations, a solution of the equations represented by the tableau is not possible unless N variables are specified. The goal of any linear programming technique is to specify the value of N of the variables in such a manner as to maximize (or minimize) the value of the objective, f.

One solution of Tableau 1 is when $x_1 = x_2 = \ldots = x_n = 0$. In that case, the whole left side of the tableau is of no consequence and the left side starting with the s_1 column forms a Basic Feasible Solution (BFS) which is

$$s_1 = b_1 \tag{11.30}$$
$$s_2 = b_2 \tag{11.31}$$
⋮
$$s_i = b_i \tag{11.32}$$
⋮
$$s_m = b_m \tag{11.33}$$

$$f = 0 \tag{11.34}$$

This is only one of many possible BFSs. The BFS is shaded here and throughout this chapter for clarity.

Other BFSs may be obtained by selecting any of the non-zero coefficients as a pivot and applying the Gauss-Jordan elimination procedure. For example, if coefficient a_{i2}, is selected as the pivot, Tableau 2 would be generated from Tableau 1. The coefficients in Tableau 2 are primed to indicate they are not the same as the corresponding coefficients in Tableau 1. Also, coefficients have been added to the tableau to replace the values of one and zero that changed. The new BFS is $s_1 = b_1'$, $s_2 = b_2'$, … $s_i = b_i'$, $s_m = b_m'$, f = 0.

Tableau 2											
x_1	x_2	...	x_n	s_1	s_2	...	s_i	...	s_m	f	RHS
a'_{11}		...	a'_{1n}	1	0	...	r_{1i}	...	0	0	b_1
a'_{21}		...	a'_{2n}	0	1	...	r_{2i}	...	0	0	b'_2
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a'_{i1}	1	...	a'_{in}	0	0	...	r_{ni}		0	0	b'_i
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a'_{m1}		...	a'_{mn}	0	0	...	r_{mi}	...	1	0	b'_m
$-c'_1$		...	$-c'_n$	0	0	0	r_{0i}	0	0	1	b'_0

This new BFS is an improvement over the first BFS since the value of the objective variable is now b'_0 rather than zero. This discovery leads to the formal statement of the Simplex Method that maximizes f.

1) Find the column with the most negative coefficient in the objective function row. This is the pivot column.

2) Divide each coefficient in the pivot column into the element in the column farthest right. Note the resulting quotient but do not change any values in the tableau. The row yielding the smallest non-negative number is the pivot row. The pivot is the intersection of the pivot column and pivot row.

3) Apply the Gauss-Jordan Elimination Procedure about the pivot.

4) Repeat steps 1 through 3 until no negative coefficients remain in the objective function row. The optimum solution is the remaining BFS.

The rationale for step (1) is to enter into the BFS the variable that has the greatest favorable impact on the objective function. Step (2) is performed to determine the equation that has the severest constraint on the production of the identified most-favorable variable. After all, the quotient is the amount of the favorable variable that each constraint will allow to be produced. Limiting production to the most severely limited constraint assures that no resource (RHS) will ever go negative – a physically impossible condition. Step (3) is simply the mathematical process of entering the favorable variable into the BFS. Step (4) continues the entire process until no adjustment in production can improve the objective. Example 3 illustrates the method for a sample problem. If the Simplex Method were applied to a problem solved by the Graphical Method each simplex cycle may be thought of as essentially moving the BFS from one vertex to an adjacent vertex.

The difference between maximizing and minimizing an objective is simply the sign of the objective function. Simplex maximizes f, but if the objective function is written in terms of –f and –f is maximized then the value of f is minimized. This means that simplex may be used without modification to minimize the objective if the f term is recast as –f in the starting tableau. To demonstrate this with simplex is difficult because meaningful examples require a $\geq$ or equality constraint. This technique will be illustrated in the Big-M Method section that can accommodate such constraints.

Example 11a

The electronics firm described at the beginning of this section has 1000 employee-hours per day for the production of iPods, PDAs, and calculators. By license agreement the firm can manufacture as many as 10 calculators per day. Furthermore, each PDA, and iPod, must pass through the soldering facility that is limited to 200 employee-hours of work per day. Each PDA, requires two hours in the soldering facility and each iPod, requires five hours. Calculators do not enter the soldering group. Each iPod, PDA, and calculator requires 20, 5, and 4 employee-hours to produce. If each iPod, PDA, and calculator returns an income of $120, $50, and $60, how many of each item should the firm produce to maximize its profit?

If
> $x =$ Number of iPods
> $y =$ Number of PDAs
> $z =$ Number of calculators

then the constraint equations are

$$
\begin{aligned}
20x + 5y + 4z &\le 1000 & (11.35)\\
5x + 2y &\le 200 & (11.36)\\
z &\le 10 & (11.37)
\end{aligned}
$$

Introducing slack variables gives

$$
\begin{aligned}
20x + 5y + 4z + s_1 &= 1000 & (11.38)\\
5x + 2y + s_2 &= 200 & (11.39)\\
z + s_3 &= 10 & (11.40)
\end{aligned}
$$

The slack variables do have some significance. For example, s_1 equals the number of employee-hours not used per day. It may be zero. The value of s_2 represents the number of hours not used in the soldering facility. The number of calculators that could have been produced per day but were not is equal to s_3. The objective function is profit and is

$$f - 120x - 50y - 60z = 0. \qquad (11.41)$$

These equations yield the following tableau:

Tableau A							
x	y	z	s_1	s_2	s_3	f	RHS
20	5	4	1				1000
5	2			1			200
		1			1		10
-120	-50	-60				1	0

The BFS to the above tableau is to manufacture nothing. In that case, (i.e. $x = y = z = 0$), and the BFS is

$$s_1 = 1000, \; s_2 = 200, \; s3 = 10, \; f = 0 \qquad (11.42)$$

The value of f can be increased by increasing either the number of iPods, PDAs, or calculators produced. However, since each iPod returns an income of $120 compared to only $50 and $60 for the other two

items, its production should be given first consideration. To determine the maximum number of iPods that can be manufactured without violating any of the constraints, one must determine the equation that most severely restricts the number of iPods that can be produced. For example, Eq. (11.38), which is represented in the Tableau A by the first row, limits the maximum number of iPods at 50. That is, each unit requires 20 employee hours and there are 1000 employee hours available. Likewise, Eq. (11.39) limits the maximum number of iPods at $200/5 = 40$. Eq. (11.40) imposes no restrictions, which is to say that insofar as Eq. (11.40) is concerned, an infinite number of iPods could be produced. These numbers are the ratios obtained in step #2 of the simplex instructions. Since the lowest number is 40, the tableau will be transformed by Gauss-Jordan elimination to have a BFS with $x = 40$. That is, the x-coefficient in the second row will become the pivot. The result is Tableau B.

Tableau B

x	y	z	s_1	s_2	s_3	f	RHS
	-3	4	1	-4			200
1	2/5			1/5			40
		1			1		10
	-2	-60		24		1	4800

The BFS is now when $y = z = s_2 = 0$ and $x = 40$, $s_1 = 200$, $s_3 = 10$, and $f = 4800$. Therefore, by producing 40 iPods, the firm's income is $4800. However, since the last row in the above tableau represents the equation.

$$-2y - 60z + f = 4800 \qquad (11.43)$$

The production of either PDAs or calculators, subject to resource constraints, would increase the income. Calculators are currently the most advantageous to the objective. Row three is the most severe constraint to the maximum number of calculators that can be produced. Making the coefficient in row three column three the pivot yields Tableau C.

Tableau C

x	y	z	s_1	s_2	s_3	f	RHS
	-3		1	-4	-4		160
1	2/5			1/5			40
		1			1		10
	-2			24	60	1	5400

The BFS is now $x = 40$, $z = 10$, $s_1 = 160$, and $f = 5400$. Therefore, income is up by $600. Also, the number of employee-hours not used is down from 200 to 160. Furthermore, the negative coefficient in the last row shows that further optimization is possible by increasing PDA production from zero to the maximum number permitted by the current constraint equations. In this case, row two limits maximum PDA, production to 100 units. Making the coefficient in row two column two the pivot yields

Tableau D

x	y	z	s_1	s_2	s_3	f	RHS
15/2			1	-5/2	-4		460
5/2	1			1/2			100
		1			1		10
5				25	60	1	5600

The BFS is now y = 100, z = 10, s_1 = 460, and f = 5600. This surprising result excludes the production of any iPods and increases the total number of unused employee-hours to 460 while increasing total income to $5600. Since the last row, which represents the objective function, no longer has any negative coefficients, the current BFS is the optimal solution.

11.4 –Linear Programming Solutions by the Big-M Method
The Graphical Method allows both ≤ and ≥ constraints but is limited to two independent variables. The Simplex Method allows unlimited variables but allows only ≤ constraints. Therefore, a method is needed that will handle many variables, both ≤ and ≥ constraints, and equality constraints. Equality constraints arise in many problems ranging from producing a specified mass of alloy from sundry scrap piles at minimum cost to the distribution of a good or service that cannot be stored such as electricity in a regional power grid. There are many methods for solving such LP Problems but the first stop is the Big-M Method. Also, it seems fitting that the Big-M Method is something every Mines student should know, even though the method is named for a factor used in the method, not SDSM&T.

The first step in the Big-M Method is to convert all of the constraint equations to equalities. In the case of ≤ constraints, slacks are added as before. Slacks represent the amount of resource that *could have been used that was not used*. To make the ≥ constraints equalities, *surplus* variables are subtracted from the left side of each such equation. As with slack variables, surplus variables are also always positive or zero. They represent the amount of a resource that *was used that did not need to be used*. Of course, equality constraints need no modification. Tableau E is an example tableau after all the constraints are made equalities.

Tableau E							
x	y	z	s_1	s_2	s_3	f	RHS
20	5	4	1				2000
5	2			1			400
		1			-1		10
1	2	1					50
-20	-50	-60				1	

Tableau E is similar to the previous Simplex Method example except the third and fourth constraints are different. The third constraint now requires that at least 10 calculators are produced. The surplus value s_3 represents the number of calculators produced that are not needed to satisfy this requirement. The fourth constraint is an equality constraint that requires that the number of iPods plus twice the number of PDAs plus the number of calculators produced must equal 50 – no more and no fewer. This seems an unrealistic constraint but nevertheless provides a suitable example to illustrate the Big-M Method.

The goal is to convert Tableau E into a form that can be processed by the Simplex Method. Examination of the tableau shows there is currently no BFS.   Since the tableau consists of 7 unknowns and 5 equations, there are 2 degrees of freedom. There are no two variables that can be set to zero to form a BFS. Even the s_3 column cannot be converted into an element in the BFS, because dividing the third constraint by -1 changes the right-hand-side resource into a negative amount. This would be a violation of the basic tenet of the Simplex Method. The escape from this dilemma is provided by the Big-M Method.

The Big-M Method dictates that each ≥ constraint and equality constraint equation receives an *artificial variable*, A. These artificial variables seem – and indeed are – a violation of the mathematical equality in each constraint unless the each have a value of 0. Since the values of the independent variables keep changing as the Simplex Method proceeds towards the solution, the inclusion of artificial variables would

not change the mathematical integrity of the problem statement so long as the final solution ends with every artificial variable equaling zero. That is, if none of the artificial variables appear in the optimal BFS, then all of the artificial variables would be zero and of no consequence. The way to assure that no artificial variable appears in the optimal BFS is to make the *cost* of using each artificial variable very detrimental to the objective. These detrimental values are termed Big M's. When these changes are made, Tableau E becomes Tableau F. If the negative values in the x, y, and z columns are favorable to the objective, the large positive M's in the artificial columns are very detrimental to the objective. This will assure that the optimal solution will contain no artificial values.

Tableau F									
x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
20	5	4	1						2000
5	2			1					400
		1			-1	1			10
1	2	1					1		50
-20	-50	-60				M	M	1	

Tableau F almost contains a BFS. All that is needed is to *drive out* the Big-M's in the artificial variable columns. This is accomplished by multiplying each constraint equation containing a "1" above each artificial value by –M and adding the resulting equation to the objective equation. For example, the third constraint equation is to be multiplied by –M and the result added to the objective function so as to drive out the M in column A_1. This is repeated for each artificial variable column. This process results in Tableau G. The boxed area shows the results of the equation-by-equation additions in the objective equation that are then summarized in the last row. Tableau H is a repeat of Tableau G without the superfluous boxed row. A BFS is now present since there are nine variables, five equations, and four degrees of freedom used to set $x = y = z = s_3 = 0$. The BFS is now

s_1	= 2000,	number of total employee hours not used that could be used.	(11.44)
s_2	= 400,	number of soldering facility hours not used that could be used.	(11.45)
A_1	= 10,	number of units of the mysterious Artificial 1 item used.	(11.46)
A_2	= 50,	number of units of the mysterious Artificial 2 (11.39)item used.	(11.47)
F	= -60M,	current profit – a very serious loss because A_1 and A_2 was used.	(11.48)

The Simplex Method can now be used to find the maximum profit. Since two artificial values must be removed from the initial BFS, at least two simplex pivot cycles must be employed to eliminate the initial artificial values from the BFS. As the artificial variables leave the BFS, the Big M terms will leave the objective row. At the optimal solution, no negative values will remain and no Big-M value will appear in the RHS of the objective row as shown in Tableaus I – K.

Tableau G									
x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
20	5	4	1						2000
5	2			1					400
		1			-1	1			10
1	2	1					1		50
-20	-50	-60				M	M	1	
		-M			M	-M			-10M
-M	-2M	-M					-M		-50M
-20	-50	-60						1	-60M
-M	-2M	-2M			M				

Tableau H

x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
20	5	4	1						2000
5	2			1					400
		1			-1	1			10
1	2	1					1		50
-20 -M	-50 -2M	-60 -2M			M			1	-60M

Tableau I

x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
20	5		1		4	-4			1960
5	2			1					400
		1			-1	1			10
1	**2**				1	-1	1		40
-20 -M	-50 -2M				-60 -M	60 2M		1	600 -40M

Tableau J

x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
17.5			1		1.5	-2	-2.5		1860
4				1	-1	1	-1		360
		1			-1	1			10
0.5	1				**0.5**	-1	0.5		20
5					-35	35 +M	25 +M	1	1600

Tableau K

x	y	z	s_1	s_2	s_3	A_1	A_2	f	RHS
16	-3		1				-4		1800
5	2			1					400
1	2	1					1		50
1	2				1	-1	1		40
40	70					M	60 +M	1	3000

The optimal solution is to produce 50 calculators and no iPods or PDAs. The slack variables tell a story of tremendous unused resources: 1800 employee hours and 400 soldering facility hours. A total of 40 calculators are being produced that do not need to be produced, but it is profitable to do so. The total profit is 3000. As expected, total artificial variable use is zero.

Linear programming problems are solved using computational packages. A Microsoft Excel® program that shows step-by-step progress through the Big-M method with helpful active dialog boxes is available at http://showard.sdsmt.edu/BigM.htm.

There are several facets of the Simplex and Big-M Methods that every user of the method should know.

- Slacks and surpluses are always ≥ 0
- For every simplex pivot cycle, one variable leaves and one variable enters the BFS.
- There must be at least as many pivot cycles as there are artificial variables.
- The *area of feasible solution* in a multivariant problem is actually a hypervolume bounded by planar hypersurfaces with feasible solutions at the vertices.
- Each simplex pivot cycle moves the feasible solution from one hypervolume vertex to an adjacent vertex.
- The only difference between minimization and maximization is whether f or –f is being maximized.
- The RHS column represents resources each of which must always ≥ 0.
- The Simplex Method always enters the next currently-most-favorable variable into the BFS.
- The f column, or –f column in the case of minimization, never change during simplex and for that reason may be removed from the tableaus.
- In minimization conditions, the RHS of the objective row is often negative at the optimal solution since it is the value of -f.
- For convenience most programmers who write Big-M solution code add an artificial variable to every constraint equation whether one is needed or not.
- No artificial variable will appear in the optimal BFS.
- The use of artificial variables creates (or creates a larger) BFS hypervolume, but the optimal solution does not fall within the domain of any artificial value.

Example 11b

Repeat the Example 10a except assume the profits are costs that are to be minimized, at least 10 calculators are produced, and that production is constrained such that

$$x + 2y + z = 50 \tag{11.49}$$

Since the objective is minimum cost, $f = 120x + 50y + 60z$, the negative of cost will be maximized. When this function is placed in the proper form shown in Eq. (11.29) for insertion into a tableau, it becomes

$$120x + 50y + 60z - f = 0 \tag{11.50}$$

The starting tableau is shown in Tableau L but must have artificial variables added and Big-M's added as shown in Tableau M before a starting BFS shown in Tableau N.

Tableau L

x	y	z	s_1	s_2	s_3	-f	RHS
20	5	4	1				2000
5	2			1			400
		1			-1		10
1	2	1					50
120	50	60				1	

Tableau M

x	y	z	s_1	s_2	s_3	A_1	A_2	-f	RHS
20	5	4	1						2000
5	2			1					400
		1			-1	1			10
1	2	1					1		50
120	50	60				M	M	1	

Tableau N

x	y	z	s_1	s_2	s_3	A_1	A_2	-f	RHS
20	5	4	1						2000
5	2			1					400
		1			-1	1			10
1	2	1					1		50
120	50	60				M	M	1	
		-M			M	-M			-10M
-M	-2M	-M					-M		-50M
120	50	60			M			1	-60M
-M	-2M	-2M							

The starting BFS above is s_1 = 2000, s_2 = 400, A_1 = 10, A_2 = 50, f = -60M. Proceeding according to the Simplex Methods yields the following:

Tableau O

x	y	z	s_1	s_2	s_3	A_1	A_2	-f	RHS
17.5		1.5	1				-2.5		1875
4		-1		1			-1		350
		1			-1	1			10
0.5	1	0.5					0.5		25
95		35			M		-25	1	-1250
		-M					M		-M

Tableau P

x	y	z	s_1	s_2	s_3	A_1	A_2	-f	RHS
17.5			1		1.5	-2	-2.5		1860
4				1	-1	1	-1		360
		1			-1	1			10
0.5	1				0.5	-1	0.5		20
95					35	-35	-25	1	-1600
						M	M		

The optimal solution is y = 20, z = 10, s_1 = 1860, s_2 = 360, x = A_1 = A_2 = 0.
The minimum cost is 1600.

11.5 –Microsoft Excel Tools
Microsoft Excel® has many features that simplify the drudgery of technical computations. Two tools that are commonly overlooked are Goal Seek and Solver. This section will outline the use of these valuable tools.

Goal Seek: Goal Seek is found under the Tools Menu as shown in Figure 2a. When it is selected, the dialog box shown in Figure 2b is available.
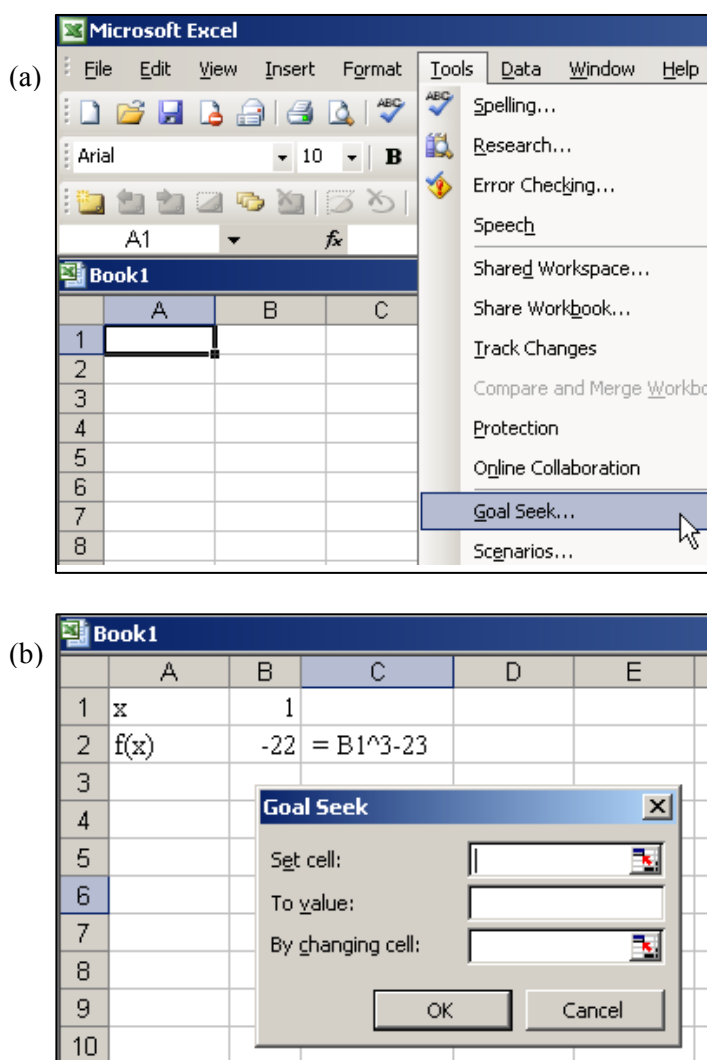

Figure 2. Accessing Goal Seek a) Opening, b) The Goal Seek Dialog Box.

Goal Seek is used to find a value of a variable that will satisfy a user-entered criterion. It will adjust one cell's value so as to change the value of another cell to a user-entered value. Figure 3a shows such a situation. Cell B2 depends on cell B1 in the way indicated (for illustration only) in Cell C2. According to the entered conditions, when the value of B1 is the cube root of 23, cell C2 will be zero. Once these selections are entered into the Goal Seek Dialog Box, Goal Seek immediately and quickly returns to cell B1 the value that satisfies the entered criterion. The returned information describes the current value of

the variable that was selected to change and the current value of the target cell. Example 10c shows the use of Goal Seek.
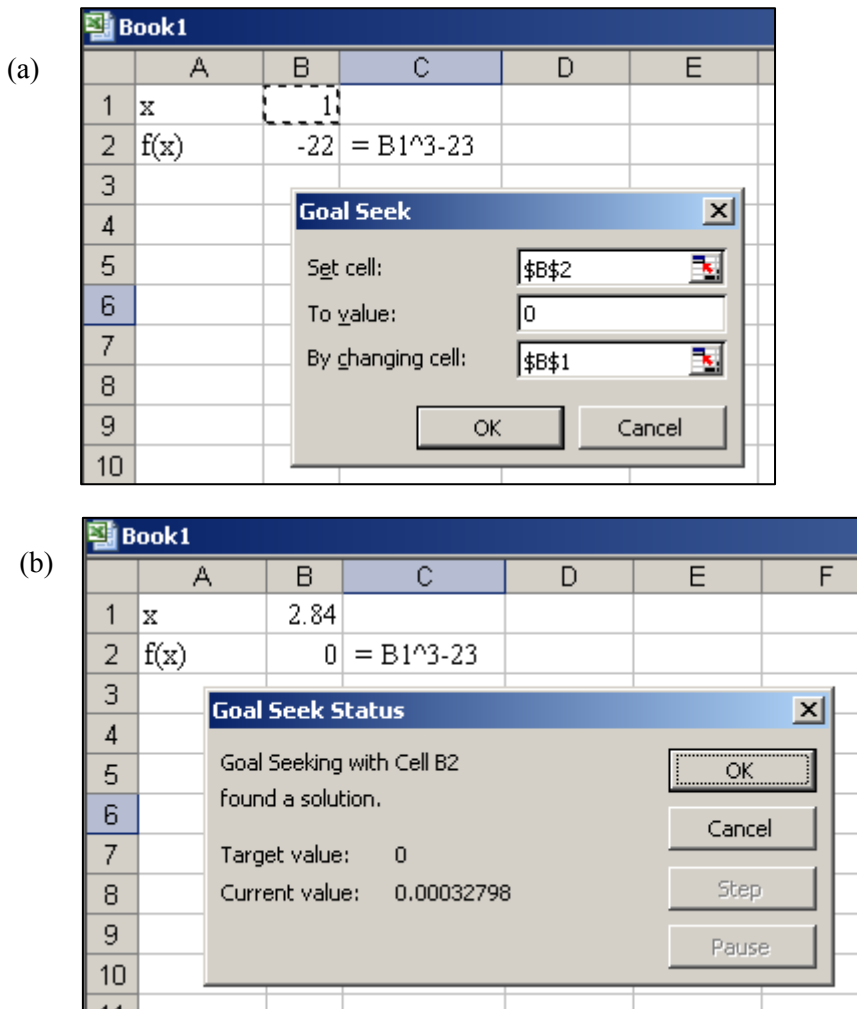
(a)

(b)

Figure 3. Use of Goal Seek a) Entering the requested information, b) the returned results.

Solver: Solver differs from Goal Seek in that it will change one, or more, values in search of a specified objective. Solver is accessed as shown in Figure 5a and consists of the dialog box shown in Figure 5b and a constraint input dialog box shown in Figure 5c. If Solver does not appear under the Tools Menu, it must be added using the Add-In Menu as shown in Figure 6a. If it is does not appear in the Add In list shown in Figure 6b, Solver needs to be installed from the Microsoft Excel Installation CD's.

Figure 5b shows the robust options available in Solver. A target worksheet cell – the value to be optimized - can be selected to be a maximum, minimum, or a specified value. The values that can be changed in order to reach the optimal value are entered as the _Changing Cells_. Constraints on the cells to be changed can be added in the _Constraints_ Panel. There must be some mathematical relationship between the _Target Cell_ and the _Changing Cells_.
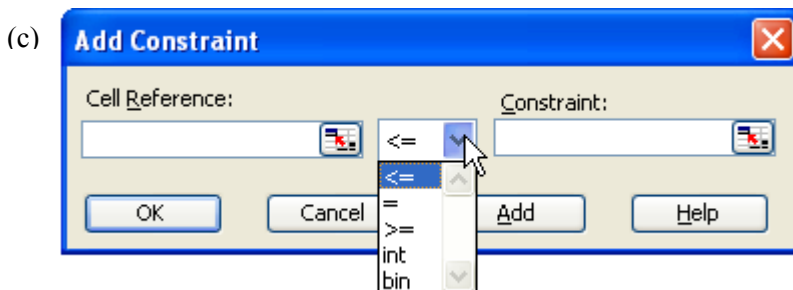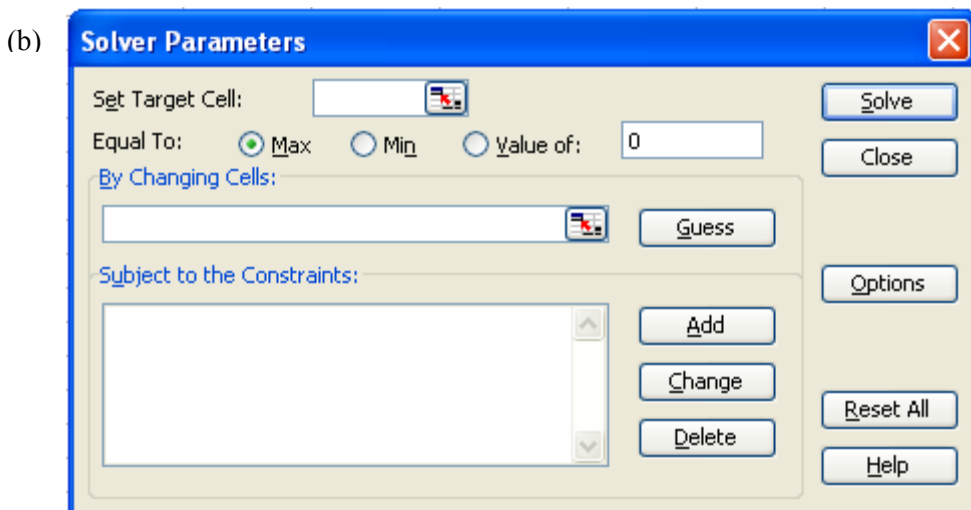
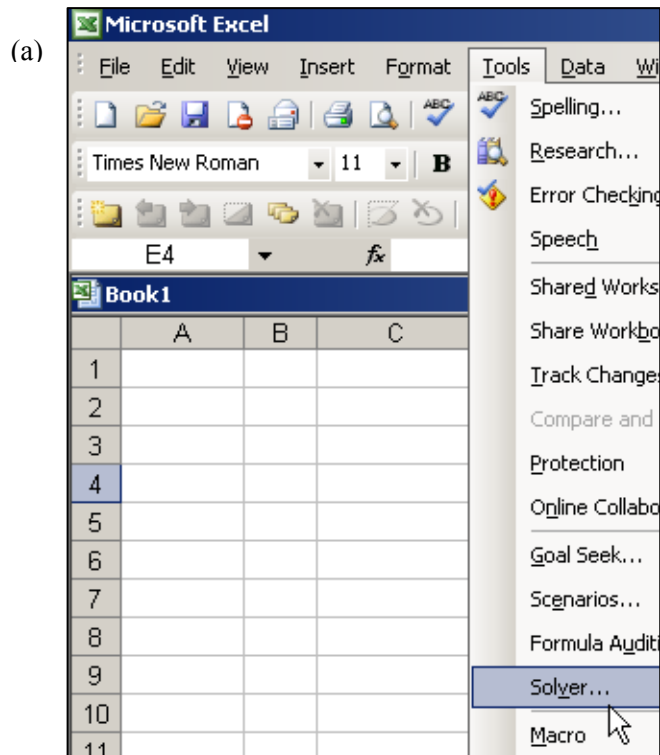Example 10d illustrates the use of Solver.

(a)

(b)

(c)

Figure 5. Solver a) Accessing, b) Dialog Box, c) Constraint Dialog Box
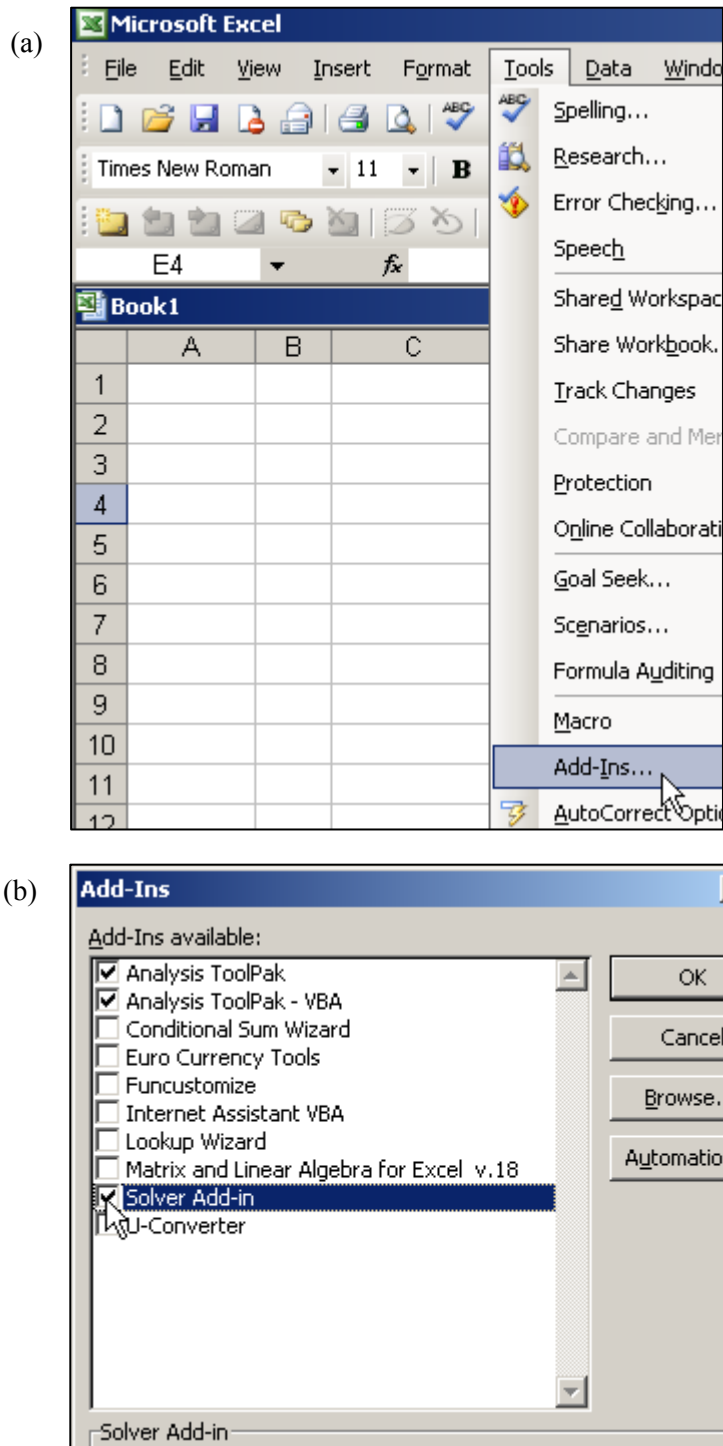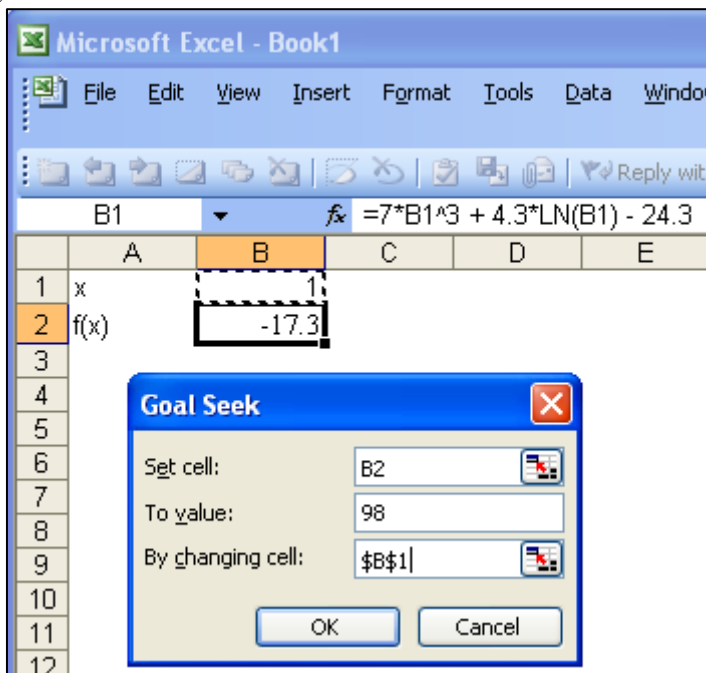
Figure 6. Adding In Solver a) Add In Menu Item, b) Selecting Solver to Add In

Example 11c

Use Goal Seek to find the value of x that makes the function $f(x) = 7x^3 + 4.3\ln(x) - 24.3$ equal to 98.
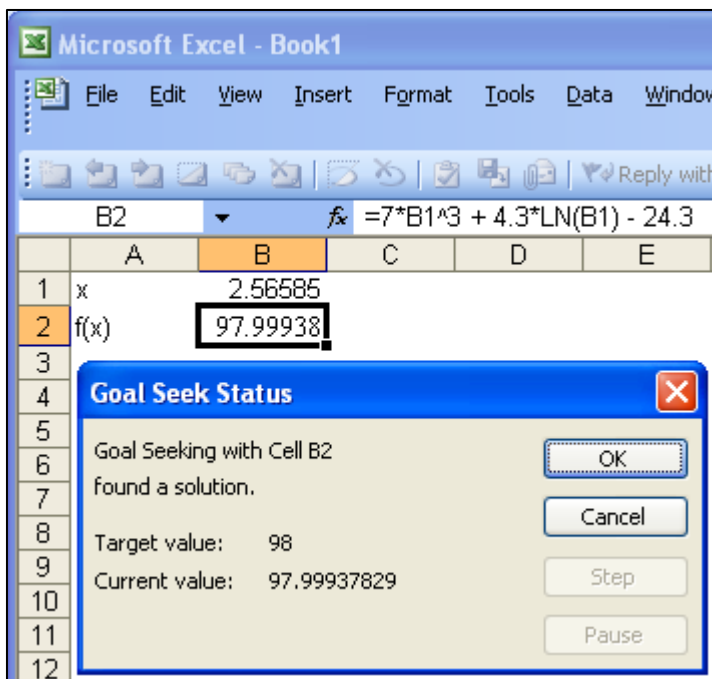
Solution

(a)

(b)

Figure 7. Solution to Example 10c

Example 11d

Use Goal Seek to solve Example 10b

Solution
Figure 8a shows the Microsoft Excel worksheet and Solver setup. The equations placed in cells A5:A7 and A8 have been pasted in to the right so the student can see how the values in column A are computed. The Changing Cells are B1:B3, which are the values of x, y, and z, the numbers of iPods, PDAs, and calculators.

Figure 8. The Microsoft Excel Worksheet and Solver Setup

In addition to the constraint equations normally entered into the Big-M Method , the Solver constraints includes a requirement that numbers of iPods, PDAs, and calculators cannot be negative. The Big-M and Simplex Methods are inherently designed to assure this never occurs, but Solver has no such limitations

so it must be explicitly stated. Otherwise, Solver will try to make the most negative numbers of the most-detrimental-to-the-objective product so as to free up resources for the more favorable use of resources. Figure 9 shows the solver solution to Example 10d.

Figure 9. Microsoft Solver Solution to Example 10d.

11.6 – Data Adjustment

Data adjustment is a method of rationalizing measured tabular data so that it conforms to mathematical relationships known to exist among the data. For example, Figure 10 shows a material processing circuit. Plant operators may periodically measure the flow rates for each streams into and out of each of the four unit operations as tabulated in Table 3. Each unit operation has material flows in and out of it. Over time, the mass of material into each operation must equal the mass out; otherwise, mass would be accumulating in a unit operation over time. Since there are four unit operations, there are four equations that must be satisfied. They are

$$A + B \ = \ C + D \tag{11.51}$$

$$C \ = \ B + E \tag{11.52}$$

$$E + F \ = \ G + H \tag{11.53}$$

$$H \ = \ F + I \tag{11.54}$$

If each column in Table 3 is averaged to get summary data for a report, one quickly discovers that Eqs. (11.51) - (11.54) are not observed. There are many reasons for this ranging from short term variations in flow rates of each stream to measurement errors. What is needed is a method that will arrive at the *best* summary values of each flow stream. The method for accomplishing this is called Data Adjustment.

No matter what values are selected for summary values, there is always going to be errors somewhere in the fit to the measured data. Mathematicians, engineers, and scientist all agree that the definition of best
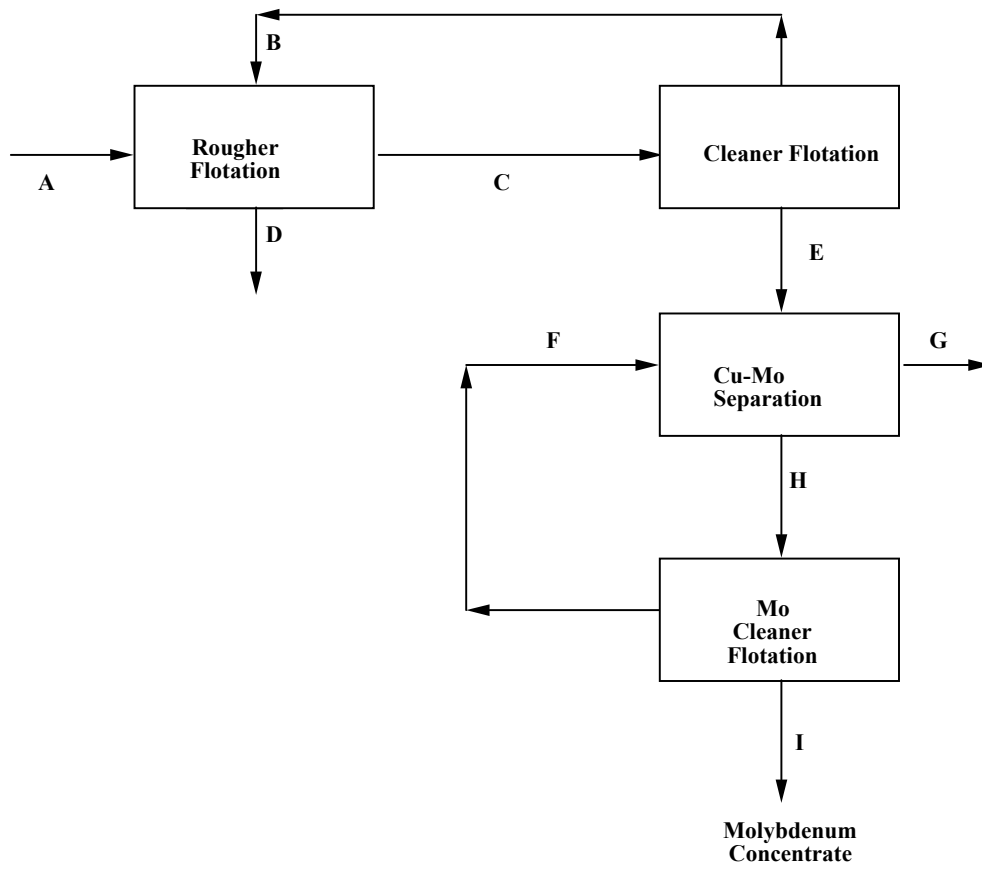
Figure 10. A Material Flow Circuit

Table 3. Measured Flow Rates for the Material Flow Circuit

A	B	C	D	E	F	G	H	I
5050	255	4200	1000	4040	395	810	3575	3180
5140	260	4375	1050	4100	410	810	3740	3300
4960	240	4200	990	3960	395	790	3600	3120
4720	230	3700	900	3810	370	720	3300	3060
5200	260	4390	1090	4120	410	830	3179	3300

values: the values that minimize the sum of the square of all each measured value and its corresponding best value. For the above problem, this is computed as follows:

$$f = \sum_{i=1}^{n}\left[\left(A_i - \overline{A}\right)^2 + \left(B_i - \overline{B}\right)^2 + \cdots \left(I_i - \overline{I}\right)^2\right] \tag{11.55}$$

Where the bar indicate the *best* values of each flow rate and n the total rows of data. The squaring of each term assures all errors (positive or negative) always contribute to the measure of *best* rather than canceling.   The best values are easily determined using Microsoft Excel Solver as shown in Example 11e.

Example 11e

Use Solver to find the best values for each flow in Table 3.

Solution
Table 4 shows the Microsoft Excel worksheet setup to compute the sum of the sum of the square of the errors for arbitrarily selected starting values for each flow rate. The first row contains names of each flow stream. The second row contains the arbitrarily-selected starting values of the independent variables. The bold values are dependent on the other, independent variables in accordance with Eqs. (11.51) - (11.54). Other combinations of dependent variables could have been selected and computed from these four constraining equations, but any combination has yields the same result.

The next five rows of the table are the measured data also shown in Table 4. The lower block of numbers contains the square of each measured value minus the corresponding column's first row values, which are to be changed to minimize the sum of all the squares of the difference appearing at the bottom of the table. Solver is used to search the independent values of A, B, C, F, and H so as to minimize the sum of the square of the differences. This yields the least-squares fit of the data while satisfying the constraints in Eqs. (11.51) - (11.54). Figure 11 shows the Solver setup and Table 5 shows the results of enacting it. There is no need to constrain all the flow rates to be ≥ 0 since there is no scenario in which making flow rates negative would improve the least-square objective.

Table 4. Data Adjustment Error Determination

A	B	C	D	E	F	G	H	I
1	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**0**
5050	255	4200	1000	4040	395	810	3575	3180
5140	260	4375	1050	4100	410	810	3740	3300
4960	240	4200	990	3960	395	790	3600	3120
4720	230	3700	900	3810	370	720	3300	3060
5200	260	4390	1090	4120	410	830	3179	3300

Diff's ^2

3.E+07	6.E+04	2.E+07	1.E+06	2.E+07	2.E+05	7.E+05	1.E+07	1.E+07
3.E+07	7.E+04	2.E+07	1.E+06	2.E+07	2.E+05	7.E+05	1.E+07	1.E+07
2.E+07	6.E+04	2.E+07	1.E+06	2.E+07	2.E+05	6.E+05	1.E+07	1.E+07
2.E+07	5.E+04	1.E+07	8.E+05	1.E+07	1.E+05	5.E+05	1.E+07	9.E+06
3.E+07	7.E+04	2.E+07	1.E+06	2.E+07	2.E+05	7.E+05	1.E+07	1.E+07

Sum of
Diff's ^2 4.E+08
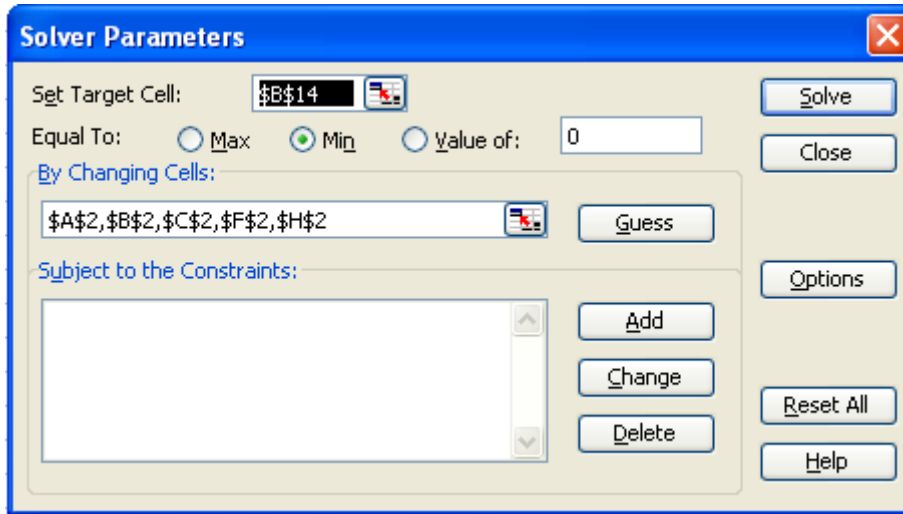
Figure 11. Solver Setup for Data Adjustment

Table 5. Optimized Data Adjustment Values

A	B	C	D	E	F	G	H	I
4999	**222**	**4200**	**1021**	**3977**	**354**	**810**	**3521**	**3167**
5050	255	4200	1000	4040	395	810	3575	3180
5140	260	4375	1050	4100	410	810	3740	3300
4960	240	4200	990	3960	395	790	3600	3120
4720	230	3700	900	3810	370	720	3300	3060
5200	260	4390	1090	4120	410	830	3179	3300

Diff's ^2

3.E+03	1.E+03	2.E-01	5.E+02	4.E+03	2.E+03	8.E-02	3.E+03	2.E+02
2.E+04	1.E+03	3.E+04	8.E+02	2.E+04	3.E+03	8.E-02	5.E+04	2.E+04
1.E+03	3.E+02	2.E-01	1.E+03	3.E+02	2.E+03	4.E+02	6.E+03	2.E+03
8.E+04	6.E+01	2.E+05	1.E+04	3.E+04	3.E+02	8.E+03	5.E+04	1.E+04
4.E+04	1.E+03	4.E+04	5.E+03	2.E+04	3.E+03	4.E+02	1.E+05	2.E+04

Sum of
Diff's ^2 8.E+05

11.7 –Regression Analysis

Regression analysis is used to obtain a best fit using the least squares concept. Data adjustment is a specialized application of regression analysis but the most common use is curve fitting. Most students have curve fitting tools on their handheld calculators. They may also use the *Add Trendline* feature of Microsoft Excel®. These tools are very useful but limit the user to select from a limit number of functions such as polynomial, power, and logarithmic. Few, if any, of these tools would allow one to find the optimal coefficients to an equation such as

$$y = ax^2 + b\ln(x) + c \qquad\qquad\qquad (11.56)$$

When a set of many (x, y) data pairs is known. Solver will allow the determination of the Least-Squares Fit as shown in Example 11f.

Example 11f

Use Solver to find the best fit coefficients for the function

$$y = ax^2 + b\ln(x) + c \tag{11.57}$$

Given

x	f(x)
1.0	5.000
1.6	1.529
2.7	5.006
4.0	1.613
13.0	0.435
27.0	-0.295

Solution

Table 6 shows the Microsoft Excel worksheet setup to compute the sum the square of the difference between the measured data (in column 2) and the computed value of the function using the current values of the coefficients a = b = c = 1 (in column 3).   Solver is use to minimize the sum of the square of the errors by changing the values of a, b, and c. Figure 12 shows the Solver setup and Table 7 is the optimal solution but still it is not perfect since the sum of the squares is not zero.

Table 6. Starting Sum of the Square of the Errors.

a	1
b	1
c	1

x	f(x)	f(x)_calc	Error Squared
1.0	5.0000	2.00	9.000
1.6	1.5300	4.03	6.250
2.7	5.0067	9.28	18.29
4.0	1.6137	18.39	281.3
13.0	0.4351	172.56	29629
27.0	-0.2958	733.30	538157
		568100	**Sum**

Solver Parameters

Set Target Cell: $D$17

Equal To: ○ Max   ● Min   ○ Value of: 0

By Changing Cells:

$B$6:$B$8   Guess

Subject to the Constraints: Options

                                               Add

                                               Change

                                               Reset All

                                               Delete

                                               Help

Solve   Close
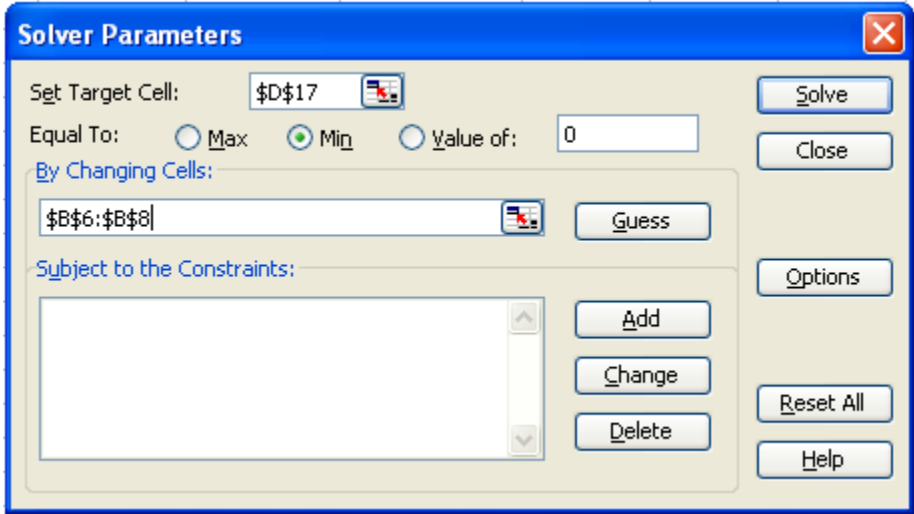
Table 7. Curve Fit Results f(x) = 0.00035 -1.481ln(x) +4.301

a	0.00035
b	-1.481
c	4.301

x	f(x)	f(x)_calc	Error Squared
1.0	5.000	4.311	0.475
1.6	1.529	3.617	4.359
2.7	5.006	2.845	4.669
4.0	1.613	2.267	0.428
13.0	0.435	0.570	0.018
27.0	-0.295	-0.345	0.003
		9.952	**Sum**