

**South Dakota School of Mines and Technology**  
**Department of Mathematics and Computer Sciences**

Math 374

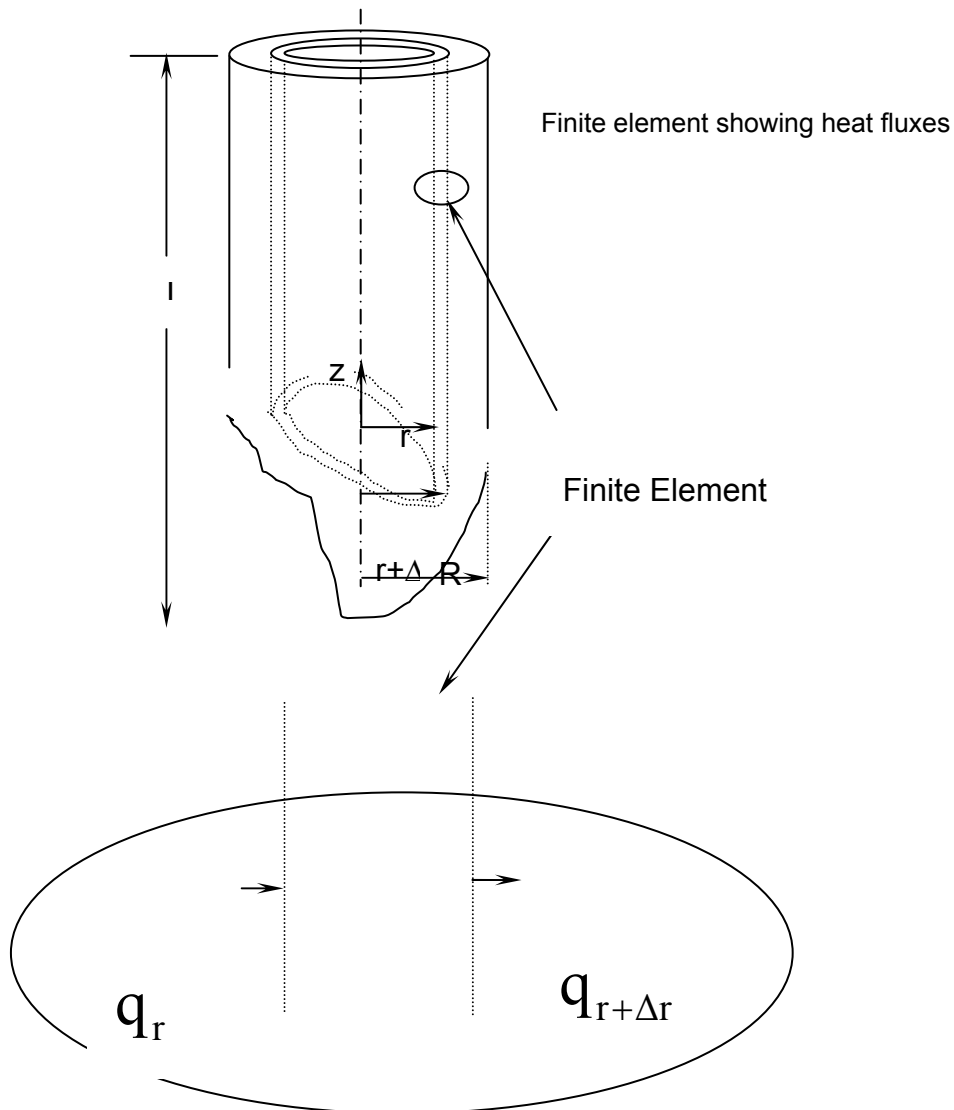
HQ 1 Retake

Feb 15, 2001

1. **Derive the Heat Equation for a one-dimensional, cylindrical coordinate, unsteady state heat conduction problem (i.e. – 1D USS HT Cyl).**

*As with all differential equation derivations, this one follows the same five steps:*

1. *Draw a sketch.*
2. *Establish an incremental element that has an incremental thickness in each direction of change.*



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3. Perform a balance on the increment for the extensive quantity of interest (force, heat, etc.).
4. Divide through by all incremental terms ( $\Delta X$ ,  $\Delta y$ ,  $\Delta t$ , etc.) and take the limit as each goes to zero to obtain differential terms.
5. Substitute for any flux terms ( $q$ ,  $\tau$ ) the fundamental law that relates these fluxes to intensive property gradients.

1 & 2 The direction of change is  $r$ . Heat Balance

$$\text{In} - \text{Out} + \text{Gen} = \text{Acc}$$

$$[2\pi L((rq)_r - (rq)_{r+\Delta r}) + 0]\Delta t = 2\pi r L \Delta r \rho C_p (T_{t+\Delta t} - T_t)$$

4. Divide by  $2\pi L \Delta r \Delta t$

$$\frac{(rq)_r - (rq)_{r+\Delta r}}{\Delta r} = r \rho C_p \frac{(T_{t+\Delta t} - T_t)}{\Delta t}$$

In the limit as each delta term approaches 0

$$-\frac{\partial rq_r}{\partial r} = r \rho C_p \frac{\partial T}{\partial t}$$

5. Substitute Fourier's Law of Heat Conduction:  $q_r = -k \frac{\partial T}{\partial r}$

$$\alpha \frac{\partial (rq_r)}{\partial r} = r \frac{\partial T}{\partial t}$$

Expanding the first term and dividing by  $r$  gives

$$\alpha \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = \frac{\partial T}{\partial t}$$

$$\text{where } \alpha = \frac{k}{\rho C_p}$$

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- 2 Sketch a spreadsheet solution to determine temperature profiles in a one-dimensional rectilinear solid as a function of position and time (1D USS HT). The solid's initial temperature is 0 and the end temperatures are both 100 for  $t \geq 0$ . Assume the solid of interest is 20 cm thick. Use 10 increments. The thermal diffusivity is  $0.125 \text{ cm}^2/\text{sec}$ . Use the maximum permissible time step. Show
- $\Delta x$ ,
  - $\Delta t$ ,
  - initial condition,
  - the boundary conditions, and
  - the pertinent equations for computation. You may indicate "Fills" to conserve effort writing.

Answer: (active Excel Spreadsheet)

	x										
time	0	2	4	6	8	10	12	14	16	18	20
0	100	0	0	0	0	0	0	0	0	0	100
16	100	50	0	0	0	0	0	0	0	50	100
32	100	50	25	0	0	0	0	0	25	50	100
48	100	63	25	13	0	0	0	13	25	63	100
64	100	63	38	13	6	0	6	13	38	63	100
80	100	69	38	22	6	6	6	22	38	69	100
96	100	69	45	22	14	6	14	22	45	69	100
112	100	73	45	30	14	14	14	30	45	73	100
128	100	73	51	30	22	14	22	30	51	73	100
144	100	76	51	37	22	22	22	37	51	76	100

3. Write the finite difference approximation for the following:

BONUS

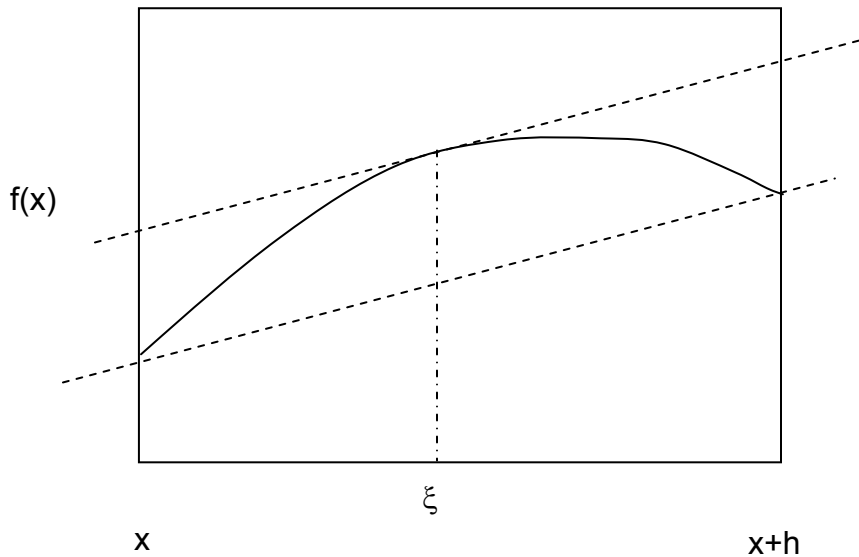
a) Forward $\frac{\partial T}{\partial t}$	$\approx \frac{T_{t+\Delta t} - T_t}{\Delta t}$	O(h <sup>N</sup> ), N= <u>1</u>
b) Central $\frac{\partial T}{\partial t}$	$\approx \frac{T_{t+\Delta t} - T_{t-\Delta t}}{2\Delta t}$	O(h <sup>N</sup> ), N= <u>2</u>
c) Backward $\frac{\partial T}{\partial t}$	$\approx \frac{T_t - T_{t-\Delta t}}{\Delta t}$	O(h <sup>N</sup> ), N= <u>1</u>

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d) Central  $\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{x-\Delta x} - 2T_x - T_{x+\Delta x}}{\Delta x^2}$

$O(h^N)$ ,  $N = \underline{\underline{2}}$

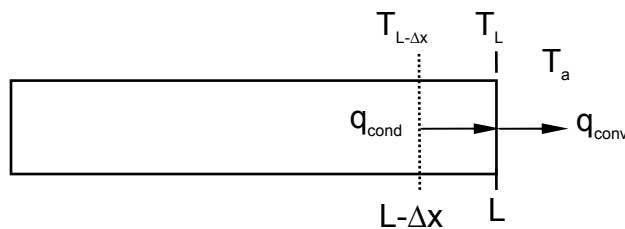
4. Indicate on the sketch below where the value of  $\xi$  lies that satisfies the Mean Value Theorem of Derivatives.



The value of  $x$  where the slope is  $\frac{f(x+h) - f(x)}{\Delta x}$

5. The solid bar below is conducting along its axis while heat is being lost by convection from the ends. Derive an equation showing the temperature  $T_L$  as a function of  $T_{L-\Delta x}$ , and  $T_a$ .

Note:  $q_{conv} = h(T_L - T_a)$  and  $q_{cond} = -k(dT/dx)$



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Answer:

$$q_{Cond} = q_{Conv}$$

$$-k \frac{T_L - T_{L-\Delta x}}{\Delta x} = h(T_L - T_a)$$

$$T_L = \frac{hT_a + \frac{k}{\Delta x} T_{L-\Delta x}}{h + \frac{k}{\Delta x}}$$